Stochastic Variety Innovation In a Growth Model

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Abstract

In endogenous growth models based on the invention of new varieties of goods, an innovation process is assumed to be *deterministic*. This note will show that this assumption can in fact be derived from a *stochastic* innovation process in many industries.

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1 Introduction

In the endogenous growth literatrue, technological innovations created through profit-seeing research activities are modelled in two forms: (i) expanding the variety of innovative goods (see Grossman and Helpman (1991, ch.3), Jones (1995), and Romer (1990)), and (ii) improving their quality (see Aghion and Howitt (1992) and Grossman and Helpman (1991, ch. 4), Segerstrom, Anant and Dinopoulos (1990)). They represent different types or trajectories of technological progress, e.g. the creation of microprocessors, and the improvement of their performance.

Endogenous growth models based on expanding variety in the form of creating imperfectly substitutable goods typically assume the following R&D technology:

$$\dot{n}(t) = \frac{l(t)\, n(t)^{\phi}}{a}, \qquad a > 0, \qquad 1 \geq \phi. \tag{1}$$

In (1), $\dot{n}(t)$ is the number of newly created varieties (i.e. technological innovations) during an infinitesimal time interval dt, $l(t)$ is the number of research workers, and the knowledge stock is equated to $n(t)^{\phi}$ which is the number of existing varieties at time t, raised to the power of ϕ. Equation (1) implies that technological breakthroughs follow a *deterministic* process where innovation ($\dot{n} > 0$) is 100% guaranteed once l workers are employed.[1]

In marked contrast, "quality-ladder" models based on quality innovation assume that innovation follows a *stochastic* process in which some research projects succeed but most fail. Indeed, Aghion and Howitt (1992, p.326) stressed this plausible assumption in comparison with the models of expanding variety, by noting that the latter "involve no un-

[1] More precisely, the "innovation production function" of a firm k is $\dot{n}^k(t) = \frac{l^k(t)n(t)^{\phi}}{a}$ where $l^k(t)$ is the number of workers and $\dot{n}^k(t)$ is innovations achieved by this firm. (1) is obtained by summing over k, i.e. $\dot{n}(t) = \sum_k \dot{n}^k(t)$ and $l(t) = \sum_k l^k(t)$.

certainty." In the literature, therefore, there is a dichotomy in the nature of technological progress: quality innovation is stochastic, but variety innovation is deterministic. However, this dominant modelling approach is at odds with the fact that virtually all research activities are characterized by a high degree of uncertainty. Costs of a given R&D project, the length of time required, whether results can be commercially meaningful, etc are typically uncertain and difficult to predict. The literature failed so far to justify the dichotomous modelling assumptions regarding the degree of uncertainty.

Motivated by this observation, this note will demonstrate that R&D technology (1) can in fact be derived from a stochastic process of variety invention taking place in a large number of industries. This result enables us to interpret variety innovation in the models of expanding variety as being generated in a stochastic environment as is the case in the quality-ladders models. This removes a dichotomy in modelling approach towards the uncertain nature of technical progress.

2 Stochastic Variety Innovation

To present our argument, we use the standard R&D-based growth model of Grossman and Helpman (1991, Ch.3) (hereafter G&H), maintaining the same notation (the time argument is suppressed where it is obvious) and the same assumptions unless otherwise stated.

G&H assume that the instantaneous utility of a representative consumer is $u = \log D$ where D is a Dixit-Stiglitz type consumption index:

$$D = \left[\int_0^n x(j)^\alpha \, dj \right]^{1/\alpha}, \qquad 1 > \alpha > 0 \tag{2}$$

where $x(j)$ denotes consumption of a variety good j. Technological advance is represented by an increase in n, e.g. the introduction of computers, cars, airplanes, TVs, videos into the economy.

We modify the consumption index (2) as

$$D = \left[\int_0^1 \sum_{j=0}^{n(i)} x_j(i)^\alpha \, di \right]^{1/\alpha}. \tag{3}$$

The integral in (2) is replaced with the summation in (3) with $j = 0, 1, 2 \ldots n(i)$ still denoting varieties.[2] In addition, we have added the integral with $i \in (0,1)$ denoting the individual manufacturing industries, in which R&D will be conducted. Thus, $x_j(i)$ is consumption of variety j in industry i.

A difference between these two specifications is that innovation is occurring in a single industry in (2), whereas (3) captures the observation that innovation takes place in many industries in the economy. Moreover, since R&D is assumed to be stochastic in (3), the number of varieties $n(i)$ differs for each i, depending upon uncertain outcome of R&D activity.

The demand function for the differentiated product (associated with (3)) is

$$x_j(i) = \frac{p_j(i)^{-\frac{1}{1-\alpha}}}{\int_0^1 \sum_{j=0}^{n(i')} p_j(i')^{-\frac{\alpha}{1-\alpha}} \, di'} \tag{4}$$

where $p_j(i)$ is the price of $x_j(i)$ and consumption expenditure is normalized. Notice that the demand function (4) has the price elasticity of $-1/(1-\alpha)$. Thus, given that one unit of differentiated products is produced with one worker, the monopoly price of variety products, the demand for each variety good and its associated profit are

$$p(i) \equiv p = \frac{w}{\alpha}, \qquad x_j(i) \equiv x = \frac{1}{p}, \qquad \pi(i) \equiv \pi = \frac{1-\alpha}{n} \tag{5}$$

[2] The summation starts from 0, i.e. goods $x_0(i)$ are assumed to be available at $t = 0$. This is a technical assumption required to sharpen the main result below.

respectively, where w is wage and

$$n = \int_0^1 n(i)\, di \tag{6}$$

is the total number of varieties in the economy. Note that the consumption index (2) also gives rise to equations (5).[3] Note that the first and third equations in (5) are equivalent to equations (3.10) and (3.11) of G&H (p.50).

$v(i)$ denotes the present value of future profit flows which a local monopoly firm will earn. It is determined by the following "no-arbitrage" condition

$$\pi(i) + \dot{v}(i) = \rho v(i) \tag{7}$$

where ρ denotes consumers' rate of time preference that is equal to the rate of interest. To achieve $v(i)$, entrepreneurs invest in R&D.

Any research firm k that uses $l^k(i)$ of workers in industry i will succeed in generating a new variety product with a Poisson arrival rate of $l^k(i)\, n^\phi / a$ where $a > 0$. This assumption emphasizes uncertainty inherent in research activities, as in quality-ladders models. Note also that the knowledge stock is equated to $n(t)^\phi$ rather than $n(i,t)^\phi$, implying that knowledge spillovers occur across industries. Firm k chooses $l^k(i)$ to maximize $v(i)\, l^k(i)\, n^\phi / a dt - w l^k(i)\, dt$ during a time interval dt. Free entry leads to

$$\frac{wa}{n^\phi} \geq v(i) \equiv v, \qquad \text{with equality whenever } l^k > 0. \tag{8}$$

In symmetric equilibrium which is indeed achieved, the industry-wide arrival rate of a new variety is

$$\frac{l n^\phi}{a}, \qquad \text{where } l \equiv l(i) = \sum_k l^k(i). \tag{9}$$

[3] Note that consumption expenditure is normalized to one, following G&H.

Given (9), we now calculate the total number of varieties $n(t) = \int_0^1 n(i,t)\,di$. Since innovation occurs with an arrival rate of (9), the probability of each industry experiencing exactly s innovations at time t is given by the Poisson density

$$\frac{z(t)^s e^{-z(t)}}{s!} \qquad \text{where } z(t) = \int_0^t \frac{l(\tau)\,n(\tau)^\phi\,d\tau}{a}. \tag{10}$$

Moreover, there is a continuum of industries, in each of which $n(i)$ rises by one. Thus, the law of large numbers implies that $n(t)$ is equivalent to the average number of innovations across industries, i.e.[4]

$$\begin{aligned} n(t) &= \sum_{s=0}^{\infty} \frac{z(t)^s e^{-z(t)}}{s!} s = z(t)\,e^{-z(t)} + \frac{z(t)^2 e^{-z(t)}}{2!} + \frac{z(t)^3 e^{-z(t)}}{3!} + \cdots \\ &= z(t) \sum_{s=0}^{\infty} \frac{z(t)^s e^{-z(t)}}{s!} = z(t). \end{aligned} \tag{11}$$

Differentiating (11) with respect to t yields

$$\dot{n}(t) = \frac{l(t)\,n(t)^\phi}{a}. \tag{12}$$

This is identical to (1) despite the fact that variety innovation is now governed by a stochastic process. The critical requirement here is that stochastic innovation is occurring in a large number of industries.

Now, assuming $\phi = 1$, (12) implies that the total research workers is given by $a\dot{n}/n$, so that full-employment of workers requires

$$\frac{a\dot{n}}{n} + \frac{1}{p} = L \tag{13}$$

where L is the total number of workers. This condition is identical to equation (3.23) of G&H (p.59). Using equations (5), (7), (8) and (13), one can easily verify that

$$\frac{\dot{n}}{n} \equiv g = (1-\alpha)\frac{L}{a} - \alpha\rho \tag{14}$$

[4] This calculation uses the fact that $j = 0$ at time $t = 0$, as assumed above.

in steady state. (14) is identical to equation (3.28) of G&H (p.61).

For $1 > \phi$, long-run growth is not sustained, given no population. Once we introduce a growth rate of population, denoted by λ, one can easily establish that $\dot{n}/n = \lambda/(1-\phi)$ in the long run. For details, see Jones (1995).

3 Concluding Remarks

This note has demonstrated that the deterministic technology of variety innovation assumed in growth models can be derived from a more general form of stochastic innovation in many industries. Our approach can be easily generalized to any growth models based on R&D technology (1).

References

[1] Aghion, P., and P. Howitt, 1992, A model of growth through creative destruction, Econometrica 60, 323-351.

[2] Grossman, G., and E. Helpman, 1991, Innovation and Growth in the Global Economy, MIT Press, Cambridge, MA.

[3] Jones, C., 1995, R&D-based models of economic growth, Journal of Political Economy 103, 759-784.

[4] Romer, P., 1990, Endogenous technological change, Journal of Political Economy 98, Part II, S71-S102.

[5] Segerstrom, P.S., T.C.A.Anant and E. Dinopoulos (1990) "A Schumpeterian Model of the Product Life Cycle." *American Economic Review*, 80, 1077-1092.